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Answer to a Question of R. Brauer*

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Brauer [*1*] has asked whether a finite group G is determined by its character table, together with the knowledge of which classes of G are nth powers of which other classes (for all integers n). The answer is no, even for p-groups. We shall prove:

THEOREM. *For any prime $p \geqslant 5$, there are two groups of exponent p and order p^7 whose character tables are isomorphic, but whose Lie rings (for their descending central series) are not isomorphic. A fortiori, the groups are not isomorphic.*

If $p \mid n$, then the identity class is the nth power of every class. If $p \nmid n$, then taking nth powers of classes is dual to taking conjugates of characters, which are determined by the character table. So our groups answer Brauer's question.

We begin by constructing the Lie algebras of our groups. These algebras depend upon a matrix A. In Lemma 1, we show that isomorphism of these algebras depends upon projective equivalence of the matrix A. Then we construct the groups. In Lemma 2, we show that all groups corresponding to nonsingular A have the same character tables. Finally we need only produce two nonsingular matrices A which are not projectively equivalent.

Let L be the free nilpotent Lie algebra of class 3 on three generators u_1, u_2, u_3 over a finite field k. Then $L = L^1 \oplus L^2 \oplus L^3$ is a direct sum of its subspaces L^i spanned by the commutators of weight $i = 1, 2, 3$ in the u's. Of course, L^1 has u_1, u_2, u_3 as a basis. For L^2, we take the basis:

$$v_1 = [u_2, u_3], \qquad v_2 = [u_3, u_1], \qquad v_3 = [u_1, u_2].$$

Then L^3 is spanned by the nine elements $[u_i, v_j]$, $i, j = 1, 2, 3$. The Jacobi identity for u_1, u_2, u_3 is:

$$[u_1, v_1] + [u_2, v_2] + [u_3, v_3] = 0. \tag{1}$$

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By Witt's formula [*2*] the dimension of L^3 is $(3^3 - 3)/3 = 8$. Hence:

Equation (1) is the only relation among the $[u_i, v_j]$ *in* L^3. (2)

Let $A = (a_{ij})$ be any nonzero 3×3 matrix with entries in k such that $\operatorname{tr} A = 0$. By (2), the linear functional:

$$\lambda_A : \sum_{i,j} r_{ij}[u_i, v_j] \to \sum_{i,j} r_{ij}a_{ij}, \qquad \text{for} \qquad r_{ij} \in k \tag{3}$$

is well-defined on L^3. Let H_A be the kernel of this linear functional. Then $0 \oplus 0 \oplus H_A$ is an ideal in L. Let L_A be the quotient Lie algebra:

$$L_A = L^1 \oplus L^2 \oplus L_A^3,$$

where $L_A^3 = L^3/H_A$. The dimension of L_A is clearly 7.

LEMMA 1. *The graded Lie algebra* L_A *is determined to within isomorphism by the projective equivalence class of the matrix* A, *and vice versa.*

Proof. Since the Lie product is alternating, we may identify L^2 uniquely with the space $L^1_\wedge L^1$ in the Grassman algebra of L^1 in such a way that $[x, y]$ corresponds to $x_\wedge y$ for all $x, y \in L^1$. Let l be any basis element for the space $L^1_\wedge L^1_\wedge L^1$. Define the linear transformation S of L^1 into the dual space L^{2*} of L^2 by:

$$x_\wedge y = Sx(y) \cdot l, \qquad \text{for all} \qquad x \in L^1, \qquad y \in L^2 = L^1_\wedge L^1. \tag{4}$$

The linear transformation S is clearly determined to within a scalar multiple by L_A (or just L!). If we choose $l = u_{1\wedge}\, u_{2\wedge}\, u_3$, then:

$$Su_i = v_i^* \qquad \text{for} \qquad i = 1, 2, 3 \tag{5}$$

where v_1^*, v_2^*, v_3^* is the dual basis to v_1, v_2, v_3.

Suppose $L_A = L^1 \oplus L^2 \oplus L_A^3$ is given. Choose a basis element l' for L_A^3. Define the linear transformation T_A of L^1 into L^{2*} by:

$$[x, y] = T_A x(y) \cdot l' \in L_A^3 \qquad \text{for} \qquad x \in L^1, \quad y \in L^2. \tag{6}$$

Again, the linear transformation T_A is determined to within a scalar multiple by L_A.

If $a_{ij} \neq 0$, and $l' = a_{ij}^{-1} \cdot [u_i, v_j] \in L_A^3$, then, for any $g, h = 1, 2, 3$:

$$[u_g, v_h] = \frac{a_{gh}}{a_{ij}} [u_i, v_j] \text{ in } L_A^3.$$

So (6) implies:

$$T_A u_g = \sum_h a_{gh} v_h^*, \qquad \text{for} \qquad g = 1, 2, 3. \tag{7}$$

The linear transformation $R_A = S^{-1}T_A : L^1 \to L^1$ is determined to within scalar multiple by L_A. By (5) and (7), its matrix with respect to the basis u_1, u_2, u_3 is a scalar multiple of A. Its matrix with respect to any other basis would therefore be equivalent to a scalar multiple of A, i.e., projectively equivalent to A. So L_A determines the projective equivalence class of A. Clearly any isomorphic graded Lie algebra would determine the same projective equivalence class.

Let B be another nonzero 3×3 matrix with entries in k and trace zero. Suppose B is projectively equivalent to A. Then we may choose a basis u_1', u_2', u_3' for L^1 and an appropriate transformation R_B whose matrix with respect to this basis is A. Let σ^1 be the linear transformation of L^1 onto L^1 sending u_i' onto u_i for $i = 1, 2, 3$. The extension of σ^1 to an isomorphism of Grassman algebras defines a linear transformation σ^2 of $L^2 = L^1_{\wedge}L^1$ onto L^2 such that

$$\sigma^2[x, y] = [\sigma^1 x, \sigma^1 y], \qquad \text{for all} \qquad x, y \in L^1. \tag{8}$$

Since the transformation S is independent of A, the identity $\sigma^1 R_B = R_A \sigma^1$ implies $T_B = \sigma^{2*} T_A \sigma^1$, where σ^{2*} is the dual map to σ^2 and an appropriate choice of T_B (from among scalar multiples) is made. Let $l' \in L_A^3$ satisfy (6), and let $l'' \in L_B^3$ satisfy the corresponding equation for T_B. Map L_B^3 onto L_A^3 by σ^3 so that $\sigma^3 l'' = l'$. Then for $x \in L^1$, $y \in L^2$:

$$\begin{aligned}[\sigma^1 x, \sigma^2 y] &= T_A \sigma^1 x(\sigma^2 y) \cdot l' \\ &= \sigma^{2*} T_A \sigma^1 x(y) \cdot \sigma^3 l'' \\ &= T_B x(y) \cdot \sigma^3 l'' \\ &= \sigma^3[x, y],\end{aligned} \tag{9}$$

where the first Lie product is in L_A and the second is in L_B.

By (8) and (9), the map $\sigma = \sigma^1 \oplus \sigma^2 \oplus \sigma^3$ is an isomorphism of L_B onto L_A. So the projective equivalence class of A determines L_A to within isomorphism.

Let p be a rational prime $\geqslant 5$. Let G be the free nilpotent group of class 3 and exponent p on three generators. Then the Lie algebra

$$(G/[G, G]) \oplus ([G, G]/[G, G, G]) \oplus [G, G, G]$$

is isomorphic to our free Lie algebra L over $k = Z_p$. (This follows easily from P. Hall's formula [3] and the fact that the free nilpotent group on r generators has the free Lie ring on r generators as its Lie ring.) We identify these two algebras. Then each of our subspaces H_A of L^3 becomes a subgroup of $[G, G, G] = L^3$. Since $[G, G, G] \subseteq$ center of G, this subgroup is normal. Let G_A be the quotient group G/H_A. Then G_A is nilpotent of class 3 having L_A as its Lie algebra (for the descending central series), LEMMA 2. *If* $\det A \neq 0$, *and* ψ *is an irreducible character of* G_A *then either:*

(a) $\psi(g) = \psi_1(g[G_A, G_A, G_A])$, *for all* $g \in G_A$ *and some irreducible character* ψ_1 of $G_A/[G_A, G_A, G_A] \simeq G/[G, G, G]$, *or*

$$\text{(b)} \qquad \begin{aligned} \psi(g) &= p^3\lambda(g) \quad &\text{if} \quad & g \in [G_A, G_A, G_A] \\ &= 0 \quad &\text{if} \quad & g \notin [G_A, G_A, G_A] \end{aligned}$$

for some linear character $\lambda \neq 1$ *of* $[G_A, G_A, G_A]$.

Proof. If ψ is trivial on $[G_A, G_A, G_A] = L_A^3$, it is given by (a). Any other ψ must appear in an induced character λ^*, where $\lambda \neq 1$ is a linear character of L_A^3. The subgroup $[G_A, G_A]$ is abelian. Let μ be an extension of λ to a linear character of $[G_A, G_A]$. We claim:

$$\textit{If the conjugate character } \mu^g \textit{ equals } \mu, \quad \textit{for some } g \in G_A, \textit{ then } g \in [G_A, G_A]. \tag{10}$$

Otherwise $g \cdot [G_A, G_A] = u \neq 0$ in L^1. Since A is nonsingular, it is clear from (3), that some element $v \in L^2$ satisfies $[u, v] \neq 0$ in L_A^3. If $g' \in [G_A, G_A]$ is chosen so that $g'[G_A, G_A, G_A] = v$, then $[g, g'] = [u, v] \neq 0$ in $L_A^3 = [G_A, G_A, G_A]$.

By construction, L_A^3 is a group of order p. So $\lambda([g, g']) \neq 1$. Hence:

$$\mu^g(g') = \mu(gg'g^{-1}) = \mu([g, g'] \cdot g') = \lambda([g, g'])\, \mu(g') \neq \mu(g').$$

Therefore: $\mu^g \neq \mu$.

Statement (10) clearly implies that the induced character μ^* is irreducible of degree p^3. It appears in λ^*. So $\lambda^* = p^3\mu^* + \cdots$. But λ^* has degree p^6. So $\lambda^* = p^3\mu^*$. Therefore $\psi = \mu^*$ is described by (b).

Our theorem is now trivial. There certainly exist two 3×3 matrices A, B over Z_p, with zero traces and nonzero determinants, which are not projectively equivalent (e.g., let A be the companion matrix to the polynomial $X^3 - 1$, and B the companion matrix to $(X + 2)(X - 1)^2$). By Lemma 1 the groups G_A, G_B have nonisomorphic Lie algebras L_A, L_B. They have exponent p, since G does, and order p^7, since $\dim L_A = \dim L_B = 7$. By lemma 2, their character tables are isomorphic.

REFERENCES

1. BRAUER, R. "Lectures on Modern Mathematics." Saaty, ed. Vol. 1, p. 138. Wiley, New York, 1963.
2. HALL, JR., M. "The Theory of Groups," p. 169. Macmillan, New York, 1959.
3. Hall, JR., M. "The Theory of Groups," p. 182, Corollary 12.3.1. Macmillan, New York, 1959.